Example 4. Having a column of Trapezoid Rule approximations, the corresponding Simpson's Rule approximations are easily found. For example, if $n = 8$, we have

$$S(8) = \frac{4T(8) - T(4)}{3} \approx 0.26423805546593.$$

The table also shows the absolute errors in the approximations. The Simpson's Rule errors decrease much more quickly than the Trapezoid Rule errors. By careful inspection, you will see that the Simpson's Rule errors decrease with a clear pattern: Each time n is doubled (or Δx is halved), the errors decrease by a factor of approximately 16, which makes Simpson's Rule a more efficient and accurate method.

Table 7.6

n	$T(n)$	$S(n)$	Error $T(n)$	Error $S(n)$
4	0.25904504019141		0.00520	
8	0.26293980164730	0.26423805546593	0.00130	0.00000306
16	0.26391564480235	0.26424092585404	0.000325	0.000000192
32	0.26415974044777	0.26424110566291	0.0000814	0.0000000120
64	0.26422077279247	0.26424111690738	0.0000203	0.000000000750
128	0.26423603140581	0.26424111761026	0.00000509	0.0000000000469

Related Exercises 31–38

QUICK CHECK 4 Compute the approximate factor by which the error decreases in Table 7.6 between $S(16)$ and $S(32)$ and between $S(32)$ and $S(64)$.

Errors in Numerical Integration

A detailed analysis of the errors in the three methods we have discussed goes beyond the scope of the book. We state without proof the standard error theorems for the methods and note that Examples 3, 4, and 6 are consistent with these results.

THEOREM 7.2 Errors in Numerical Integration

Assume that f'' is continuous on the interval $[a, b]$ and that k is a real number such that $|f''(x)| < k$ for all x in $[a, b]$. The absolute errors in approximating the integral $\int_a^b f(x)\,dx$ by the Midpoint Rule and Trapezoid Rule with n subintervals satisfy the inequalities

$$E_M \leq \frac{k(b-a)}{24}(\Delta x)^2 \quad \text{and} \quad E_T \leq \frac{k(b-a)}{12}(\Delta x)^2$$

respectively, where $\Delta x = (b - a)/n$.

Assume that $f^{(4)}$ is continuous on the interval $[a, b]$ and that K is a real number such that $|f^{(4)}(x)| < K$ on $[a, b]$. The error in approximating the integral $\int_a^b f(x)\,dx$ by Simpson's Rule with n subintervals satisfies the inequality

$$E_S \leq \frac{K(b-a)}{180}(\Delta x)^4.$$

The absolute errors associated with the Midpoint Rule and Trapezoid Rule are propo- tional to $(\Delta x)^2$. So, if Δx is reduced by a factor of 2, the errors decrease roughly by a fac- tor of 4, as seen in Example 4. Simpson's Rule is a more accurate method; its error proportional to $(\Delta x)^4$, which means that if Δx is reduced by a factor of 2, the errors de- crease roughly by a factor of 16, as seen in Example 6. Computing both the Trapezoid Rule and Simpson's Rule together, as shown in Example 6, is a powerful method that produce accurate approximations with relatively little work.

SECTION 7.6 EXERCISES

Review Questions

1. If the interval $[4, 18]$ is partitioned into $n = 28$ subintervals of equal width, what is Δx?
2. Explain geometrically how the Midpoint Rule is used to approximate a definite integral.
3. Explain geometrically how the Trapezoid Rule is used to approximate a definite integral.
4. If the Midpoint Rule is used on the interval $[-1, 11]$ with $n = 3$ subintervals, at what x-coordinates is the integrand evaluated?
5. If the Trapezoid Rule is used on the interval $[-1, 9]$ with $n = 5$ subintervals, at what x-coordinates is the integrand evaluated?
6. State how to compute the Simpson's Rule approximation $S(2n)$ if the Trapezoid Rule approximations $T(2n)$ and $T(n)$ are known.

Basic Skills

T 7–10. Absolute and relative error *Compute the absolute and relative errors in using c to approximate x.*

7. $x = \pi$; $c = 3.14$
8. $x = \sqrt{2}$; $c = 1.414$
9. $x = e$; $c = 2.72$
10. $x = e$; $c = 2.718$

T 11–14. Midpoint Rule approximations *Find the indicated Midpoint Rule approximations to the following integrals.*

11. $\int_2^{10} 2x^2\,dx$ using $n = 1, 2$, and 4 subintervals
12. $\int_1^9 x^3\,dx$ using $n = 1, 2$, and 4 subintervals
13. $\int_0^1 \sin \pi x\,dx$ using $n = 6$ subintervals
14. $\int_0^1 e^{-x}\,dx$ using $n = 8$ subintervals

T 15–18. Trapezoid Rule approximations *Find the indicated Trapezoid Rule approximations to the following integrals.*

15. $\int_2^{10} 2x^2\,dx$ using $n = 2, 4$, and 8 subintervals
16. $\int_1^9 x^3\,dx$ using $n = 2, 4$, and 8 subintervals
17. $\int_0^1 \sin \pi x\,dx$ using $n = 6$ subintervals
18. $\int_0^1 e^{-x}\,dx$ using $n = 8$ subintervals

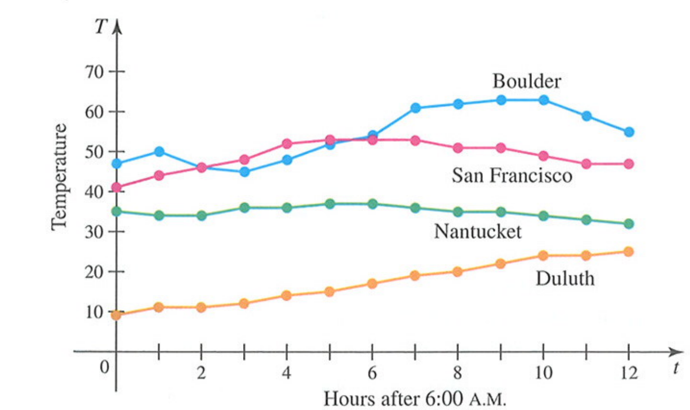
T 19. Midpoint Rule, Trapezoid Rule and relative error Find the Midpoint and Trapezoid Rule approximations to $\int_0^1 \sin \pi x\,dx$ using $n = 25$ subintervals. Compute the relative error of each approximation.

T 20. Midpoint Rule, Trapezoid Rule and relative error Find the Midpoint and Trapezoid Rule approximations to $\int_0^1 e^{-x}\,dx$ using $n = 50$ subintervals. Compute the relative error of each approximation.

T 21–26. Comparing the Midpoint and Trapezoid Rules *Apply the Midpoint and Trapezoid Rules to the following integrals. Make a table similar to Table 7.4 showing the approximations and errors for $n = 4, 8, 16$, and 32. The exact values of the integrals are given for computing the error.*

21. $\int_1^5 (3x^2 - 2x)\,dx = 100$
22. $\int_{-2}^6 \left(\frac{x^3}{16} - x\right) dx = 4$
23. $\int_0^{\pi/4} 3 \sin 2x\,dx = \frac{3}{2}$
24. $\int_1^e \ln x\,dx = 1$
25. $\int_0^{\pi} \sin x \cos 3x\,dx = 0$
26. $\int_0^8 e^{-2x}\,dx = \frac{1 - e^{-16}}{2} \approx 0.4999999$

T 27–30. Temperature data *Hourly temperature data for Boulder, CO, San Francisco, CA, Nantucket, MA, and Duluth, MN, over a 12-hr period on the same day of January are shown in the figure. Assume that these data are taken from a continuous temperature function $T(t)$. The average temperature over the 12-hr period is $\overline{T} = \frac{1}{12}\int_0^{12} T(t)\,dt$.*



t	0	1	2	3	4	5	6	7	8	9	10	11	12
B	47	50	46	45	48	52	54	61	62	63	63	59	55
SF	41	44	46	48	52	53	53	53	51	51	49	47	47
N	35	34	34	36	36	37	37	36	35	35	34	33	32
D	9	11	11	12	14	15	17	19	20	22	24	24	25

27. Find an accurate approximation to the average temperature over the 12-hr period for Boulder. State your method.

28. Find an accurate approximation to the average temperature over the 12-hour period for San Francisco. State your method.

29. Find an accurate approximation to the average temperature over the 12-hr period for Nantucket. State your method.

30. Find an accurate approximation to the average temperature over the 12-hr period for Duluth. State your method.

T **31–34. Trapezoid Rule and Simpson's Rule** *Consider the following integrals and the given values of n.*

a. *Find the Trapezoid Rule approximations to the integral using n and 2n subintervals.*

b. *Find the Simpson's Rule approximation to the integral using 2n subintervals. It is easiest to obtain Simpson's Rule approximations from the Trapezoid Rule approximations, as in Example 6.*

c. *Compute the absolute errors in the Trapezoid Rule and Simpson's Rule with 2n subintervals.*

31. $\int_0^1 e^{2x}\,dx;\ n = 25$

32. $\int_0^2 x^4\,dx;\ n = 30$

33. $\int_1^e \frac{1}{x}\,dx;\ n = 50$

34. $\int_0^{\pi/4} \frac{1}{1 + x^2}\,dx;\ n = 64$

T **35–38. Simpson's Rule** *Apply Simpson's Rule to the following integrals. It is easiest to obtain the Simpson's Rule approximations from the Trapezoid Rule approximations, as in Example 6. Make a table similar to Table 7.6 showing the approximations and errors for n = 4, 8, 16, and 32. The exact values of the integrals are given for computing the error.*

35. $\int_0^4 (3x^5 - 8x^3)\,dx = 1536$

36. $\int_1^e \ln x\,dx = 1$

37. $\int_0^\pi e^{-t}\sin t\,dt = \frac{1}{2}(e^{-\pi} + 1)$

38. $\int_0^6 3e^{-3x}\,dx = 1 - e^{-18} \approx 1.000000$

Further Explorations

39. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. The Trapezoid Rule is exact when used to approximate the definite integral of a linear function.

b. If the number of subintervals used in the Midpoint Rule is increased by a factor of 3, the error is expected to decrease by a factor of 8.

c. If the number of subintervals used in the Trapezoid Rule is increased by a factor of 4, the error is expected to decrease by a factor of 16.

T **40–43. Comparing the Midpoint and Trapezoid Rules** *Compare the errors in the Midpoint and Trapezoid Rules with n = 4, 8, 16, and 32 subintervals when they are applied to the following integrals (with their exact values given).*

40. $\int_0^{\pi/2} \sin^6 x\,dx = \frac{5\pi}{32}$

41. $\int_0^{\pi/2} \cos^9 x\,dx = \frac{128}{315}$

42. $\int_0^1 (8x^7 - 7x^8)\,dx = \frac{2}{9}$

43. $\int_0^\pi \ln(5 + 3\cos x)\,dx = \pi \ln\frac{9}{2}$

T **44–47. Using Simpson's Rule** *Approximate the following integrals using Simpson's Rule. Experiment with values of n to ensure that the error is less than 10^{-3}.*

44. $\int_0^{2\pi} \frac{dx}{(5 + 3\sin x)^2} = \frac{5\pi}{32}$

45. $\int_0^\pi \frac{\cos x}{\frac{5}{4} - \cos x}\,dx = \frac{2\pi}{3}$

46. $\int_0^\pi \ln(2 + \cos x)\,dx = \pi \ln\left(\frac{2 + \sqrt{3}}{2}\right)$

47. $\int_0^\pi \sin 6x \cos 3x\,dx = \frac{4}{9}$

Applications

T **48. Period of a pendulum** A standard pendulum of length L swinging under only the influence of gravity (no resistance) has a period of

$$T = \frac{4}{\omega}\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2\sin^2\varphi}}$$

where $\omega^2 = g/L$, $k^2 = \sin^2(\theta_0/2)$, $g \approx 9.8\ \text{m/s}^2$ is the acceleration due to gravity, and θ_0 is the initial angle from which the pendulum is released (in radians). Use numerical integration to approximate the period of a pendulum with $L = 1$ m that is released from an angle of $\theta_0 = \pi/4$ rad.

T **49. Arc length of an ellipse** The length of an ellipse with axes of length $2a$ and $2b$ is

$$\int_0^{2\pi} \sqrt{a^2\cos^2 t + b^2\sin^2 t}\,dt.$$

Use numerical integration and experiment with different values of n to approximate the length of an ellipse with $a = 4$ and $b = 8$.

T **50. Sine Integral** The theory of diffraction produces the sine integral function $\text{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt$. Use the Midpoint Rule to approximate Si (1) and Si (10). (Recall that $\lim_{x\to 0}(\sin x)/x = 1$.) Experiment with the number of subintervals until you obtain approximations that have an error less than 10^{-3}. A rule of thumb is that if two successive approximations differ by less than 10^{-3}, then the error is usually less than 10^{-3}.

T **51. Normal distribution of heights** The heights of U.S. men are normally distributed with a mean of 69 in and a standard deviation of 3 in. This means that the fraction of men with a height between a and b (with $a < b$) inches is given by the integral

$$\frac{1}{3\sqrt{2\pi}}\int_a^b e^{-[(x-69)/3]^2/2}\,dx.$$

What percentage of American men are between 66 and 72 inches in height? Use the method of your choice and experiment with the number of subintervals until you obtain successive approximations that differ by less than 10^{-3}.

52. Normal distribution of movie lengths A recent study revealed that the lengths of U.S. movies are normally distributed with a mean of 110 min and a standard deviation of 22 min. This means that the fraction of movies with lengths between a and b minutes (with $a < b$) is given by the integral

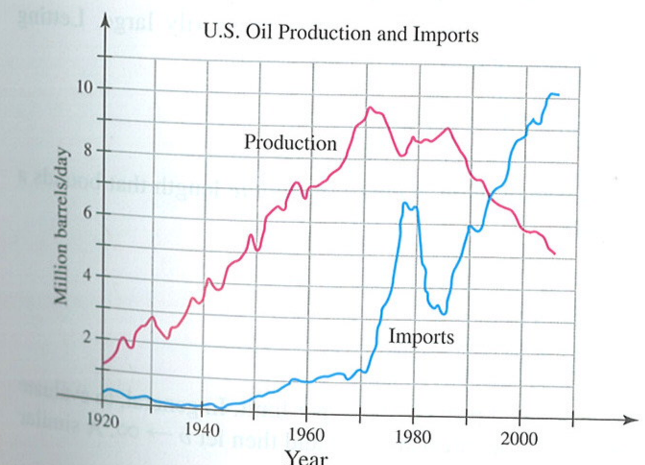
$$\frac{1}{22\sqrt{2\pi}}\int_a^b e^{-[(x-110)/22]^2/2}\,dx.$$

What percentage of U.S. movies are between 1 hr and 1.5 hr long (60–90 min)?

53. U.S. oil produced and imported The figure shows the rate at which U.S. oil was produced and imported between 1920 and 2005 in units of millions of barrels per day. The total amount of oil produced or imported is given by the area of the region under the corresponding curve. Be careful with units because both days and years are used in this data set.

a. Use numerical integration to estimate the amount of U.S. oil produced between 1940 and 2000. Use the method of your choice and experiment with values of n.

b. Use numerical integration to estimate the amount of oil imported between 1940 and 2000. Use the method of your choice and experiment with values of n.



Source: U.S. Energy Information Administration

Additional Exercises

54. Estimating error Refer to Theorem 7.2 and let $f(x) = e^{x^2}$.

a. Find a Trapezoid Rule approximation to $\int_0^1 e^{x^2}\,dx$ using $n = 50$ subintervals.

b. Calculate $f''(x)$.

c. Explain why $|f''(x)| < 18$ on $[0, 1]$, given that $e < 3$.

d. Use Theorem 7.2 to find an upper bound on the absolute error in the estimate found in part (a).

T **55. Estimating error** Refer to Theorem 7.2 and let $f(x) = \sin e^x$.

a. Find a Trapezoid Rule approximation to $\int_0^1 \sin(e^x)\,dx$ using $n = 40$ subintervals.

b. Calculate $f''(x)$.

c. Explain why $|f''(x)| < 6$ on $[0, 1]$, given that $e < 3$. (*Hint:* Graph f''.)

d. Find an upper bound on the absolute error in the estimate found in part (a) using Theorem 7.2.

56. Exact Trapezoid Rule Prove that the Trapezoid Rule is exact (no error) when approximating the definite integral of a linear function.

57. Exact Simpson's Rule Prove that Simpson's Rule is exact (no error) when approximating the definite integral of a linear function *and* a quadratic function.

58. Shortcut for the Trapezoid Rule Prove that if you have $M(n)$ and $T(n)$ (a Midpoint Rule approximation and a Trapezoid Rule approximation with n subintervals), then $T(2n) = (T(n) + M(n))/2$.

59. Trapezoid Rule and concavity Suppose f is positive and its first two derivatives are continuous on $[a, b]$. If f'' is positive on $[a, b]$, then is a Trapezoid Rule estimate of $\int_a^b f(x)\,dx$ an underestimate or overestimate of the integral? Justify your answer using Theorem 7.2 and an illustration.

60. Shortcut for Simpson's Rule Using the notation of the text, prove that $S(2n) = \frac{4T(2n) - T(n)}{3}$ for $n \geq 1$.

61. Another Simpson's Rule formula Another Simpson's Rule formula is $S(2n) = \frac{2M(n) + T(n)}{3}$ for $n \geq 1$. Use this rule to estimate $\int_1^e 1/x\,dx$ using $n = 10$ subintervals.

QUICK CHECK ANSWERS

1. 4, 6, 8, 10 **2.** Overestimates **3.** 4 and 4
4. 16 and 16